MATH 3060 Tutorial 4

Chan Ki Fung

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1. Let d_1, d_2 be two metrics on a set X. Suppose d_1 is stronger than d_2.
   (a) Let $\{x_n\}$ be a sequence in X that is convergent with respect to d_1. Show that $\{x_n\}$ converges with respect to d_2 as well.
   (b) Let Y be another metric space, and $g : Y \to X$ is a map that is continuous with respect to d_1, show that g is also continuous with respect to d_2.
   (c) Let Z be another metric space, and $f : X \to Z$ is a map that is continuous with respect to d_2, show that f is also continuous with respect to d_1.
   (d) Let $U \subset X$ be a subset that is open with respect to d_2, show that U is also open with respect to d_1.

2. (a) Let (X, d) be a metric space, show that $d' = d/(1 + d)$ is a also a metric.
   (b) Show that a sequence $\{x_n\}$ converges with respect to d if and only if it converges with respect to d'
   (c) Show by example that d' may not be stronger then d.

3. Let (X, d) be a metric space. Let $C_0(X)$ be the set of bounded continuous functions from X to $\mathbb{R}$, let ρ be the metric on $C_0(X)$ defined by $\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|$. For each $x \in X$, consider the function $l_x : X \to \mathbb{R}$ defined as $l_x(y) = d(x, y)$. Show that $l_x \in C_0(X)$, and the assignment $x \mapsto l_x$ is continuous.

4. (Bolzano-Weierstrass?) Let X be the set of real-valued continuous functions on $[-\pi, \pi]$. Let $\{f_n\}$ be a sequence in X, bounded in L_2 norm. Does $\{f_n\}$ have a Cauchy subsequence?